Math 72 7.4 Radical Expressions

Objectives

1) Add & subtract like radicals
2) Multiply radicals
3) Rationalize denominators
    - denominator with one term
    - denominator with two terms

Math 60 9.5 Adding, Subtracting, and Multiplying Radical Expressions

Objectives 1) Add or subtract radical expressions

2) Multiply radical expressions

Recall: Combine like terms
- same bases
- same exponents

① $(3x + 2y - 4) + (5x - 7y - 1)$

$= \underbrace{3x + 5x}_{\text{add coefficients only}} + 2y - 7y - 4 - 1$

$= \boxed{8x - 5y - 5}$

When we want to combine radicals by adding, they must be <u>like radicals</u>.

- Same radicands $\Longleftrightarrow$ same base
- Same index $\Longleftrightarrow$ same exponent

$\sqrt{x} = x^{1/2}$ can be combined only with other $\sqrt{x}$, not with $\sqrt{y}$ or $\sqrt[3]{x}$.

We combine like radicals by adding coefficients.

Add or subtract. Assume all variables are non-negative

② $3\sqrt{5x} + 7\sqrt{5x}$

$= \underbrace{(3+7)}_{\text{add coefficients}}\sqrt{5x}$

$= \boxed{10\sqrt{5x}}$

③ $5\sqrt[3]{11} - 8\sqrt[3]{11} + \sqrt[3]{11}$

$= (5 - 8 + 1)\sqrt[3]{11}$

$= \boxed{-2\sqrt[3]{11}}$

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Assume all variables are non-negative.

(4) $3\sqrt{20} + 8\sqrt{45}$

$= 3\sqrt{4\cdot 5} + 8\sqrt{9\cdot 5}$

index 2 $\Rightarrow$ want perfect square factors

Not like radicals $\sqrt{20}$ and $\sqrt{45}$ have the same index, but different radicands.

$20 = 4 \cdot 5 \qquad 45 = 9 \cdot 5$

$= 3\cdot\sqrt{4}\cdot\sqrt{5} + 8\sqrt{9}\cdot\sqrt{5}$ Simplify each radical.

$= 3\cdot 2\cdot\sqrt{5} + 8\cdot 3\cdot\sqrt{5}$ multiply coefficients

$= 6\sqrt{5} + 24\sqrt{5}$ <u>NOW</u> they are like radicals!

$= (6+24)\sqrt{5}$

$= \boxed{30\sqrt{5}}$

(5) $6x\sqrt{12x} - 5\sqrt{3x^3}$ Not like radicals.

$= 6x\cdot\sqrt{4\cdot 3\cdot x} - 5\sqrt{3\cdot x^2\cdot x}$ index 2 $\Rightarrow$ find perfect square factors.

$= 6x\sqrt{4}\cdot\sqrt{3x} - 5\cdot\sqrt{x^2}\cdot\sqrt{3x}$

$= 6x\cdot 2\cdot\sqrt{3x} - 5\cdot x\cdot\sqrt{3x}$ simplify perfect squares

$= 12x\cdot\sqrt{3x} - 5x\sqrt{3x}$ multiply coefficients

$= (12x-5x)\sqrt{3x}$ <u>Now</u> we have like radicals.

$= \boxed{7x\sqrt{3x}}$

$12x$ and $5x$ are like terms, so these can be subtracted.

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⑥ $2\sqrt{11} + 8\sqrt{6}$

$\boxed{\text{cannot be simplified}}$

$\sqrt{11}$ and $\sqrt{6}$ cannot be simplified because neither 11 nor 6 has a perfect sq. factor.

⑦ $\sqrt[3]{-54x^4} + 5x\sqrt[3]{2x} + x\sqrt[3]{16x}$

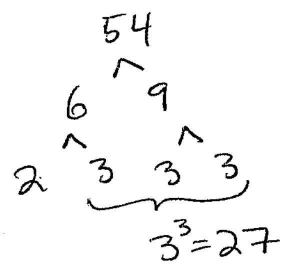
Assume all variables are non-negative.

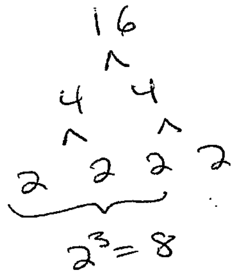
Handwriting alert!

If your handwriting is sloppy, the index 3 of the radical could migrate and become an exponent 3 on the coefficient x.

Avoid this by using extra space, or a multiplication dot.

index 3 ⟹ want perfect cube factors





$= \sqrt[3]{-27 \cdot x^3 \cdot 2x} + 5x\sqrt[3]{2x} + x\sqrt[3]{8 \cdot 2x}$

Simplify radicals than can be simplified.

$= \sqrt[3]{-27}\,\sqrt[3]{x^3}\,\sqrt[3]{2x} + 5x\sqrt[3]{2x} + x \cdot \sqrt[3]{8} \cdot \sqrt[3]{2x}$

$= -3x \cdot \sqrt[3]{2x} + 5x \cdot \sqrt[3]{2x} + 2x \cdot \sqrt[3]{2x}$

Now like radicals

$= (-3x + 5x + 2x) \cdot \sqrt[3]{2x}$

$= \boxed{4x\sqrt[3]{2x}}$

Math 60 9.5

(8) $2\sqrt{a^2b} - 5a\sqrt{b}$ NOT like radicals

Assume all variables are non-negative.

Simplify $2\sqrt{a^2b}$

index 2 $\Rightarrow$ perfect square

$= 2\sqrt{a^2} \cdot \sqrt{b}$

$= 2a\sqrt{b}$

$= 2a\sqrt{b} - 5a\sqrt{b}$ Now like radicals.

$= (2a - 5a)\sqrt{b}$

$= \boxed{-3a\sqrt{b}}$

Review: Multiply vs Add.

(9) $3\sqrt{5} + 8\sqrt{5}$

$= (3+8)\sqrt{5}$

$= \boxed{11\sqrt{5}}$

Combine like radicals

(10) $3\sqrt{5} \cdot 8\sqrt{5}$

$= 3 \cdot 8 \cdot \sqrt{5} \cdot \sqrt{5}$

$= 24 \cdot 5$

$= \boxed{120}$

Multiply and simplify.

(11) $\sqrt{6}\,(\underset{\substack{\uparrow \\ \text{1st} \\ \text{term}}}{3} - \underset{\substack{\uparrow \\ \text{2nd} \\ \text{term}}}{2\sqrt{6}})$ distribute $\sqrt{6}$.

$= 3 \cdot \sqrt{6} - 2 \cdot \sqrt{6} \cdot \sqrt{6} = 3\sqrt{6} - 2 \cdot 6 = \boxed{3\sqrt{6} - 12}$

Math 60 9.5

(12) $\sqrt[3]{4}\,(5+\sqrt[3]{2})$ distribute

$= \sqrt[3]{4}\cdot 5 + \sqrt[3]{4}\cdot\sqrt[3]{2}$

$= 5\sqrt[3]{4} + \sqrt[3]{8}$ multiply radicands.

$= \boxed{5\sqrt[3]{4} + 2}$

(13) $(8-3\sqrt{2})(5+7\sqrt{2})$ multiply by FOIL

$\underset{\text{1st term}}{8}$, $\underset{\text{2nd term}}{-3\sqrt{2}}$, $\underset{\text{1st term}}{5}$, $\underset{\text{2nd term}}{7\sqrt{2}}$

$= 8\cdot 5 + 8\cdot 7\cdot\sqrt{2} - 3\sqrt{2}\cdot 5 - 3\sqrt{2}\cdot 7\sqrt{2}$

$= 40 + \underbrace{56\sqrt{2} - 15\sqrt{2}}_{\text{like radicals}} - 21\cdot\underbrace{\sqrt{2}\cdot\sqrt{2}}_{2}$

$= 40 + (56-15)\sqrt{2} - 42$

$= 40 - 42 + 41\sqrt{2}$

$= \boxed{-2+41\sqrt{2}}$

(14) $-36 - 5\sqrt{7}$

$\boxed{\text{cannot be combined/simplified}}$

(15) $(5\sqrt{7}+\sqrt{2})^2$

$= (5\sqrt{7}+\sqrt{2})(5\sqrt{7}+\sqrt{2})$ multiply by FOIL

$= 5\cdot 5\cdot\sqrt{7}\cdot\sqrt{7} + 5\cdot\sqrt{2}\cdot\sqrt{7} + 5\cdot\sqrt{2}\cdot\sqrt{7} + \sqrt{2}\cdot\sqrt{2}$

$= 25\cdot 7 + 5\sqrt{14} + 5\sqrt{14} + 2$

$= 175 + 10\sqrt{14} + 2 = \boxed{177 + 10\sqrt{14}}$

Math 60 9.5

16 $(8-\sqrt{5})(8+\sqrt{5})$ multiply by FOIL

$= 8 \cdot 8 + 8\sqrt{5} - 8\sqrt{5} - \sqrt{5} \cdot \sqrt{5}$

$= 64 + (8-8)\sqrt{5} - 5$

$= 59 + 0\sqrt{5}$

$= \boxed{59}$

This is like a difference of squares — the middle terms add to zero.

We will capitalize on this result in 9.6.

Math 60 9.5 and 9.6 Formula in MathXL

In 9.5 and 9.6, we often multiply conjugates:

1. $(2+\sqrt{3})(2-\sqrt{3})$
2. $(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})$
3. $(3\sqrt{7}+2\sqrt{5})(3\sqrt{7}-2\sqrt{5})$

MathXL uses a formula: $(a-b)(a+b)=a^2-b^2$

or $(a+b)(a-b)=a^2-b^2$

to remind you that the FOIL process creates terms in the middle that add to 0 and disappear.

This formula is just the factoring formula for the difference of squares $a^2-b^2=(a+b)(a-b)$, only it's written backwards.

Examples:

1. $\underset{(a+b)(a-b)}{(2+\sqrt{3})(2-\sqrt{3})} = \underset{a^2}{4} \underbrace{-2\sqrt{3}+2\sqrt{3}}_{+0} \underset{-b^2}{-3} = \underset{a^2 \; (2)^2}{4} - \underset{b^2 \; (\sqrt{3})^2}{3} = \boxed{1}$

2. $\underset{(a-b)(a+b)}{(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})} = \underset{a^2}{5} \underbrace{+\sqrt{10}-\sqrt{10}}_{+0} \underset{-b^2}{-2} = \underset{a^2 \; (\sqrt{5})^2}{5} - \underset{b^2 \; (\sqrt{2})^2}{2} = \boxed{3}$

3. $\underset{(a+b)(a-b)}{(3\sqrt{7}+2\sqrt{5})(3\sqrt{7}-2\sqrt{5})} = \underset{a^2}{9\cdot 7} \underbrace{-6\sqrt{35}+6\sqrt{35}}_{+0} \underset{-b^2}{-4\cdot 5}$

$= \underset{a^2}{63} - \underset{b^2}{20} = \boxed{43}$

Math 60 9.6 Rationalizing Radical Expressions

Objectives
1) Rationalize a denominator containing one term
   - square root
   - higher-index root
2) Rationalize a denominator containing two terms
   - square roots only

Recall: A <u>rational number</u> is a number that can be written as a fraction ex: $\frac{2}{3}$, $.2$, 4, $-.\overline{6}$

and has a repeating or terminating decimal.
Key: Does not have a radical.

Recall: An <u>irrational number</u> is a number that cannot be written as a fraction. ex. $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, π, $\sqrt[3]{2}$, $\sqrt[3]{4}$, $\sqrt[4]{2}$

To <u>rationalize</u> is to re-write as an equivalent expression so that the irrational number is moved to different place.

This is a purely cosmetic change
- The entire expression is an irrational number
- When we are done, the entire expression is still the <u>same</u> irrational number.
- We are only changing how it looks
  - make denominator rational
  - make numerator irrational

Why? Because in higher-level classes (like calculus) we need to rearrange terms so we can simplify.

- How? We multiply by 1.
  But we will write 1 as a fraction, with the same radical in the numerator and in the denominator.

But first, let's review multiplying and notice whether the final answer is rational or irrational.

Math 60 9.6

Review

Multiply.
Identify if the answer is rational or irrational.

① $\sqrt{5} \cdot \sqrt{5}$

$= \sqrt{25}$

$= \boxed{5}$ $\boxed{\text{rational}}$

② $\sqrt[3]{2} \cdot \sqrt[3]{4}$

$= \sqrt[3]{8}$

$= \boxed{2}$ $\boxed{\text{rational}}$

③ $\sqrt[4]{2} \cdot \sqrt[4]{8}$

$= \sqrt[4]{16}$

$= \boxed{2}$ $\boxed{\text{rational}}$

④ $\sqrt[3]{2} \cdot \sqrt[3]{2}$

$= \boxed{\sqrt[3]{4}}$ $\boxed{\text{irrational}}$

4 is not a perfect cube, so this radical does not simplify.

multiplying a cube root by itself does NOT give a rational result. ☹

⑤ $3\sqrt{7} \cdot \sqrt{7}$

$= 3\sqrt{49}$

$= 3 \cdot 7$

$= \boxed{21}$ $\boxed{\text{rational}}$

⑥ $(1+\sqrt{2})(1-\sqrt{2})$

$= 1 \underbrace{- \sqrt{2} + \sqrt{2}}_{+0} - \sqrt{4}$ FOIL

$= 1 + 0 - 2$

$= \boxed{-1}$ $\boxed{\text{rational}}$

This is a difference of squares!
$(a+b)(a-b) = a^2 - b^2$.

Math 60 9.6

(7) $(5-\sqrt{3})(5+\sqrt{3})$

$= 25 \underbrace{+5\sqrt{3} - 5\sqrt{3}} - \sqrt{9}$

$= 25 + 0 - 3$

$= \boxed{22}$ $\boxed{\text{rational}}$

(8) $(5-\sqrt{3})^2$

$= (5-\sqrt{3})(5-\sqrt{3})$ FOIL

$= 25 \underbrace{-5\sqrt{3} - 5\sqrt{3}} - \sqrt{9}$

$= 25 - 10\sqrt{3} - 3$ combine like radicals

$= \boxed{22 - 10\sqrt{3}}$ $\boxed{\text{irrational}}$

When there are 2 terms, multiplying by itself does <u>NOT</u> give a rational result. ☹.

(9) $\sqrt{3}(5-\sqrt{3})$

$= 5\sqrt{3} - \sqrt{9}$ distribute

$= \boxed{5\sqrt{3} - 3}$ $\boxed{\text{irrational}}$

When there are 2 terms, multiplying by only the radical does <u>NOT</u> give a rational result. ☹.

<u>Definition</u>: When two binomials multiply to give a difference of squares, we say the binomials are <u>conjugates</u> of each other.

Example: $(1-\sqrt{2})$ is the conjugate of $(1+\sqrt{2})$ and vice-versa, because $(1-\sqrt{2})(1+\sqrt{2}) = 1^2 - (\sqrt{2})^2$

Example: $(5+\sqrt{3})$ is the conjugate of $(5-\sqrt{3})$ and vice-versa, because $(5+\sqrt{3})(5-\sqrt{3}) = 5^2 - (\sqrt{3})^2$

Find the conjugate of each expression.

(10) $6-\sqrt{3}$ conjugate is $\boxed{6+\sqrt{3}}$

(11) $3+\sqrt{5}$ conjugate is $\boxed{3-\sqrt{5}}$

(12) $2\sqrt{3}-4\sqrt{5}$ conjugate is $\boxed{2\sqrt{3}+4\sqrt{5}}$

Math 60 9.6

Rationalize each denominator.

(13) $\dfrac{1}{\sqrt{5}}$ notice: denominator is irrational

multiply by $1 = \dfrac{\sqrt{5}}{\sqrt{5}}$

$= \dfrac{1}{\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}}$

$= \dfrac{\sqrt{5}}{\sqrt{25}}$

$= \boxed{\dfrac{\sqrt{5}}{5}}$ notice: denominator is rational. The denominator has been rationalized.

(14) $\dfrac{\sqrt{3}}{\sqrt{32}}$

simplify $\sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$

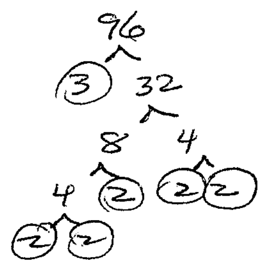
$= \dfrac{\sqrt{3}}{4\sqrt{2}}$

multiply by $1 = \dfrac{\sqrt{2}}{\sqrt{2}}$

$= \dfrac{\sqrt{3}}{4\sqrt{2}} \cdot \dfrac{\sqrt{2}}{\sqrt{2}}$

$= \dfrac{\sqrt{6}}{4 \cdot 2}$

$= \boxed{\dfrac{\sqrt{6}}{8}}$



Note: While it is legal to use $\dfrac{\sqrt{32}}{\sqrt{32}}$, it's ugly:

$\dfrac{\sqrt{3}}{\sqrt{32}} \cdot \dfrac{\sqrt{32}}{\sqrt{32}} = \dfrac{\sqrt{96}}{32} = \dfrac{4\sqrt{6}}{32} = \dfrac{\sqrt{6}}{8}$.

Math 60 9.6 Assume all variables are positive.

(15) $\dfrac{3}{2\sqrt{5x}}$

multiply by $1 = \dfrac{\sqrt{5x}}{\sqrt{5x}}$

$$= \frac{3}{2\sqrt{5x}} \cdot \frac{\sqrt{5x}}{\sqrt{5x}}$$

$$= \frac{3\sqrt{5x}}{2 \cdot 5x}$$

$$= \boxed{\frac{3\sqrt{5x}}{10x}}$$

(16) $\dfrac{5}{\sqrt[3]{2}}$

Remember: $\sqrt[3]{2} \cdot \sqrt[3]{2} = \sqrt[3]{4}$ is not rational!

We need $\sqrt[3]{2^3}$ so we are missing $\sqrt[3]{2^2} = \sqrt[3]{4}$

$$= \frac{5}{\sqrt[3]{2}} \cdot \frac{\sqrt[3]{4}}{\sqrt[3]{4}}$$

$$= \frac{5\sqrt[3]{4}}{\sqrt[3]{8}}$$

$$= \boxed{\frac{5\sqrt[3]{4}}{2}}$$

(17) $\sqrt[3]{\dfrac{5}{12}}$

Because this fraction has a radical on its denominator, its denom is not rationalized.

$$= \frac{\sqrt[3]{5}}{\sqrt[3]{12}}$$

12
6 (2)
(2) (3)

$\sqrt[3]{12} = \sqrt[3]{2^2 \cdot 3}$

need $\sqrt[3]{2}$ need $\sqrt[3]{3^2}$

$$= \frac{\sqrt[3]{5}}{\sqrt[3]{2^2 \cdot 3}} \cdot \frac{\sqrt[3]{2 \cdot 3^2}}{\sqrt[3]{2 \cdot 3^2}}$$

$$= \frac{\sqrt[3]{5 \cdot 2 \cdot 3^2}}{\sqrt[3]{2^3 \cdot 3^3}}$$

because there are no perfect cubes in the prime factors of 90, we cannot simplify $\sqrt[3]{90}$.

$$= \frac{\sqrt[3]{90}}{2 \cdot 3}$$

$$= \boxed{\frac{\sqrt[3]{90}}{6}}$$

NOTE: If you have the correct denominator but the wrong numerator, check the index of the radical. You probably didn't use the correct radicand (and assumed the correct result for denominator.)

(18) $\dfrac{10}{\sqrt[4]{2z^2}}$ Assume all variables are positive.

missing 2^3 missing z^2

notice index 4 $\Rightarrow$ We need perfect 4th powers in denom $\sqrt[4]{2^4 \cdot z^4}$

$$= \frac{10}{\sqrt[4]{2z^2}} \cdot \frac{\sqrt[4]{2^3 \cdot z^2}}{\sqrt[4]{2^3 \cdot z^2}}$$

$$= \frac{10\sqrt[4]{8z^2}}{\sqrt[4]{2^4 z^4}}$$

$$= \frac{10\sqrt[4]{8z^2}}{2z}$$

Notice $\frac{10}{2} = 5$ reduces.

$$= \boxed{\frac{5\sqrt[4]{8z^2}}{z}}$$

Math 60 9.6

(19) $\dfrac{11}{6-\sqrt{3}}$

Notice: Two terms in denominator.

Remember that multiplying a binomial by its conjugate gives a rational number.

The conjugate of $6-\sqrt{3}$ is $6+\sqrt{3}$.

$= \dfrac{11}{(6-\sqrt{3})} \cdot \dfrac{(6+\sqrt{3})}{(6+\sqrt{3})}$ multiply by $1 = \dfrac{6+\sqrt{3}}{6+\sqrt{3}}$

$= \dfrac{11(6+\sqrt{3})}{36+6\sqrt{3}-6\sqrt{3}-\sqrt{9}}$ ← Leave 11 outside because it's rational We hope to reduce. ↙ FOIL denom

$= \dfrac{11(6+\sqrt{3})}{36-3}$

$= \dfrac{11(6+\sqrt{3})}{33}$ notice $\dfrac{11}{33}$ reduces to $\dfrac{1}{3}$ $(\div 11)$

$= \boxed{\dfrac{6+\sqrt{3}}{3}}$ also equal to $\dfrac{6}{3}+\dfrac{\sqrt{3}}{3} = \boxed{2+\dfrac{\sqrt{3}}{3}}$

(20) $\dfrac{\sqrt{3}}{3+\sqrt{5}}$

The conjugate of $3+\sqrt{5}$ is $3-\sqrt{5}$

$= \dfrac{\sqrt{3}}{(3+\sqrt{5})} \cdot \dfrac{(3-\sqrt{5})}{(3-\sqrt{5})}$ — dist numerator (because $\sqrt{\ }$ outside is irrational ⇒ no chance to reduce) — FOIL denom

$= \dfrac{3\sqrt{3}-\sqrt{15}}{9+3\sqrt{5}-3\sqrt{5}-\sqrt{25}}$

$= \dfrac{3\sqrt{3}-\sqrt{15}}{9-5} = \boxed{\dfrac{3\sqrt{3}-\sqrt{15}}{4}}$

Math 60 9.6

21. $\dfrac{\sqrt{5}-2}{\sqrt{3}-\sqrt{5}}$

The conjugate of $\sqrt{3}-\sqrt{5}$ is $\sqrt{3}+\sqrt{5}$

$$= \frac{(\sqrt{5}-2)}{(\sqrt{3}-\sqrt{5})} \cdot \frac{(\sqrt{3}+\sqrt{5})}{(\sqrt{3}+\sqrt{5})}$$

← FOIL numerator

← FOIL denominator

$$= \frac{\sqrt{15}+5-2\sqrt{3}-2\sqrt{5}}{3+\sqrt{15}-\sqrt{15}-\sqrt{25}}$$

$$= \frac{5-2\sqrt{3}-2\sqrt{5}+\sqrt{15}}{3-5}$$

It's not required that you write the radicals in order, but it's nicer if you do.

$$= \frac{5-2\sqrt{3}-2\sqrt{5}+\sqrt{15}}{-2}$$

negative in denominator is not simplified

$$= \frac{-(5-2\sqrt{3}-2\sqrt{5}+\sqrt{15})}{2}$$

move negative to numerator

$$= \boxed{\frac{-5+2\sqrt{3}+2\sqrt{5}-\sqrt{15}}{2}}$$

Math 60 9.6

22. $$\frac{4\sqrt{3}+5\sqrt{2}}{2\sqrt{3}-4\sqrt{5}} \cdot \frac{2\sqrt{3}+4\sqrt{5}}{2\sqrt{3}+4\sqrt{5}}$$

$$= \frac{8\cdot 3 + 16\sqrt{15} + 10\sqrt{6} + 20\sqrt{10}}{4\cdot 3 - 16\cdot 5}$$

$$= \frac{24 + 16\sqrt{15} + 10\sqrt{6} + 20\sqrt{10}}{12-80}$$

$$= \frac{24 + 16\sqrt{15} + 10\sqrt{6} + 20\sqrt{10}}{-68}$$

$$= \frac{+2\left(12 + 8\sqrt{15} + 5\sqrt{6} + 10\sqrt{10}\right)}{-2(34)}$$

$$= \frac{-\left(12 + 8\sqrt{15} + 5\sqrt{6} + 10\sqrt{10}\right)}{34}$$

$$= \boxed{\frac{-12 - 8\sqrt{15} - 5\sqrt{6} - 10\sqrt{10}}{34}}$$